Applications of Lagrange's equations of motion :—

(1) Linear Harmonic oscillator:

Lagrange's equation of motion for one dimensional motion, say in x direction, can be written as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \text{———} \quad (1)$$

The kinetic energy of this system is

$$T = \frac{1}{2} m\dot{x}^2$$

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and potential energy is

$$V = -\int \vec{F} \cdot d\vec{x}$$

$$= -\int -kx \cdot dx$$

$$V = \frac{1}{2}kx^2 + C$$

where, C is a constant of integration and k is spring constant. If we choose the horizontal plane passing through the position of equilibrium as the reference level, then $V=0$ at $x=0$, so that $C=0$. Thus Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

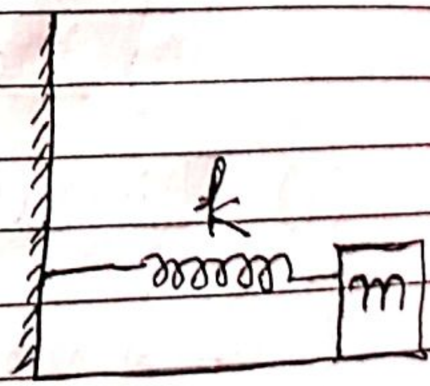

Fig. Linear harmonic oscillator

So that $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$$\frac{\partial L}{\partial x} = -kx$$

eq. (1) takes the form

$$\frac{d}{dt}(m\dot{x}) - (-kx) = 0$$

or, $m\ddot{x} + kx = 0$,

which is the required expression. This eq^n

can be put in the form $\ddot{x}+\omega^2 x=0$, where ω is the frequency of oscillation given by $\sqrt{\left(\frac{k}{m}\right)}$.

(2) <u>Simple pendulum:</u>

~~From fig.~~ The angle θ between rest position and deflected position is chosen as generalized co-ordinate. If the string is of length l, then kinetic energy is

$$T=\frac{1}{2}mv^2=\frac{1}{2}m(l\dot{\theta})^2$$

$$=\frac{1}{2}ml^2\dot{\theta}^2$$

Where, m is the mass of the bob.

In coming from position B to A, the mass has fallen freely through a vertical distance CA. Thus, potential energy is

$$V=mg(OA-OC)$$

$$=mg(l-l\cos\theta)$$

$$=mgl(1-\cos\theta)$$

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where the reference level or zero level of potential energy has been taken at a distance l below the point of suspension.

Thus, Lagrangian is

$$L = T - V$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta)$$

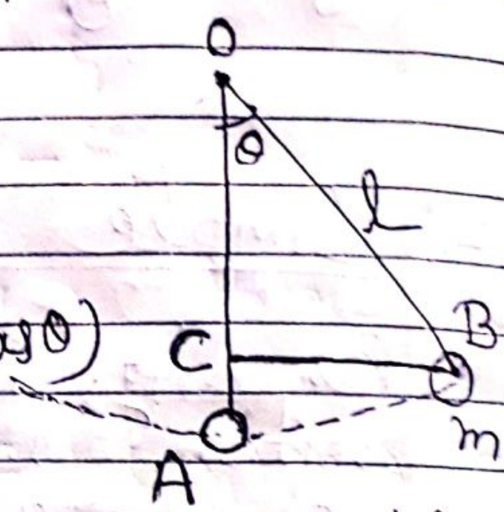

fig: Simple pendulum

so that, $\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$

and $\frac{\partial L}{\partial \theta} = -mgl \sin\theta$

Putting in Lagrange's equation.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0,$$

We get

$$\frac{d}{dt}(m l^2 \dot{\theta}) + mgl \sin\theta = 0$$

or, $m l^2 \ddot{\theta} + mgl \sin\theta = 0,$

or, $\ddot{\theta} + \frac{g}{l} \sin\theta = 0$

If the amplitude of the motion is small enough, that is $\sin\theta = \theta$, then

$$\boxed{\ddot{\theta} + \frac{g}{l}\theta = 0}$$

which is equation for angular simple harmonic motion with period

$$\boxed{T = 2\pi\sqrt{\left(\frac{l}{g}\right)}}$$

(3) ATWOOD'S MACHINE:

Atwood's machine is a holanomic conservative system with one degree of freedom and if the pulley is frictionless, constraints are scleronomous.

System consists of two masses m_1 and m_2 suspended over a frictionless pulley of radius a and connected by a flexible string of constant length l as shown in fig.

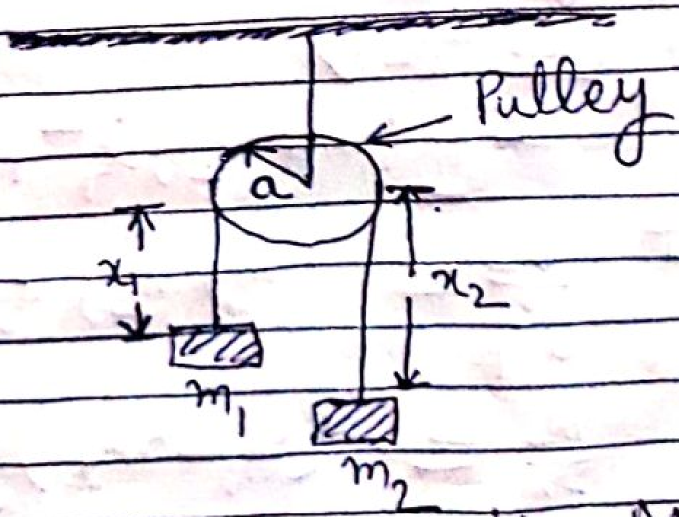

Fig. Atwood's Machine

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The configuration can easily be specified by two co-ordinates x_1 and x_2. Equation of constraint for such a system is

$$x_1 + x_2 + \pi a = l$$

$$\dot{x}_2 = -\dot{x}_1$$

So that $\ddot{x}_2 = -\ddot{x}_1$

The kinetic energy is therefore,

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2$$

and then potential energy

$$V = -m_1 g x_1 - m_2 g x_2$$

$$= -(m_1 - m_2) g x_1 + V_0,$$

where, $V_0 = -g m_2 (l - \pi a) = $ a const.

Now Lagrangian is

$$L = T - V$$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + (m_1 - m_2) g x_1 - V_0$$

so that $\frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1$

$$\frac{\partial L}{\partial x_1} = (m_1 - m_2) g,$$

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which, on being substituted in

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

gives, $\frac{d}{dt}[(m_1+m_2)\dot{x}_1] - (m_1-m_2)g = 0$

or, $$\boxed{\ddot{x}_1 = \frac{(m_1-m_2)}{(m_1+m_2)}\,g}$$

Which is desired.

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